#### Practical introduction to Agda

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"Agda is a proof assistant [...] for developing constructive proofs based on the Curry-Howard correspondence [...]. It can also be seen as a functional programming language with dependent types."

Wikipedia on Agda

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My goal here is to explain what these key concepts mean and how they combine to form Agda.

"Agda is a proof assistant [...] for developing constructive proofs based on the Curry-Howard correspondence [...]. It can also be seen as a functional programming language with dependent types."

Wikipedia on Agda

My goal here is to explain what these key concepts mean and how they combine to form Agda.

I am assuming familiarity with Haskell, or other mainstream functional programming languages.

#### Part I

# A crash course on the Curry-Howard correspondence

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#### Types as static guarantees

Types matter because they enable automated checking of certain properties.

Trivial example: map

$$map :: (\alpha \to \beta) \to [\alpha] \to [\beta]$$

Haskell tracks side-effects, so by looking at map's type, we already know that it does no IO.

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More involved example: ST

 $\begin{array}{ll} \textit{newSTRef} :: \alpha & \rightarrow \textit{ST} \ \sigma \ (\textit{STRef} \ \sigma \ \alpha) \\ \textit{readSTRef} :: \textit{STRef} \ \sigma \ \alpha & \rightarrow \textit{ST} \ \sigma \ \alpha \\ \textit{runST} & :: (\forall \ \sigma. \ \textit{ST} \ \sigma \ \alpha) \rightarrow \alpha \end{array}$ 

The parametricity of the computation passed to *runST* ensures that references don't leak

What properties can we express in types? Is this all just a collection of ad-hoc kludges exploiting lucky coincidences?



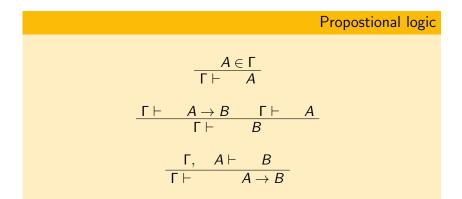
#### The simply typed lambda calculus

$$\frac{x:A\in\Gamma}{\Gamma\vdash x:A}$$

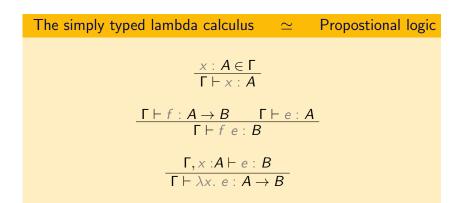
$$\frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash e : A}{\Gamma \vdash f \; e : B}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x. \ e : A \to B}$$

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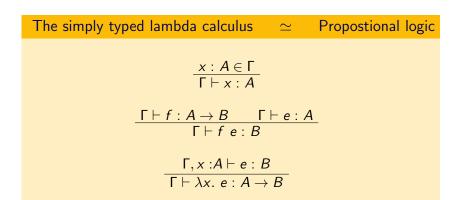


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The type inference rules of the STLC directly parallel the deduction rules of ZOL. Hence, *types*  $\simeq$  *propositions*.

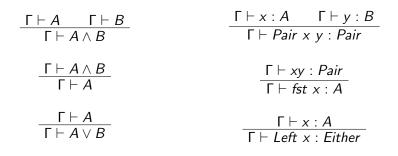


The type inference rules of the STLC directly parallel the deduction rules of ZOL. Hence, *types*  $\simeq$  *propositions*. *Terms*  $\simeq$  *proofs*, with functions corresponding to proofs that assume other properties.

## Simple extensions to ZOL

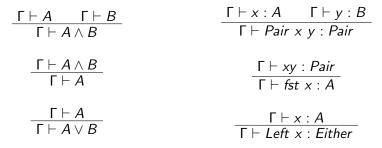
$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \land B} \\
\frac{\Gamma \vdash A \land B}{\Gamma \vdash A} \\
\frac{\Gamma \vdash A}{\Gamma \vdash A \lor B}$$

Simple extensions to ZOL and STLC



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#### Simple extensions to ZOL and STLC



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We can introduce these axioms as datatypes:

data 
$$Pair = Pair A B$$
  
fst  $(Pair x y) = x$   
snd  $(Pair x y) = y$   
data Either = Left  $A \mid Right B$ 

HM is already more expressive than these simple extensions because it offers polymorphism. We can *abstract over propositions*:

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 $\textit{const}::\alpha\to\beta\to\alpha$ 

HM is already more expressive than these simple extensions because it offers polymorphism. We can *abstract over propositions*:

*const* ::  $\alpha \rightarrow \beta \rightarrow \alpha$ 

or with parametric datatypes, introduce whole new axiom schemes:

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data Pair  $\alpha \beta = Pair \alpha \beta$ data Either  $\alpha \beta = Left \alpha \mid Right \beta$  We could regard the Haskell type checker as a proof assitant: using C-H, we can encode our propositions as types, and if the type checker accepts our definition x :: A, then we can regard A as proven.

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Two problems with this approach:

We can't express predicates

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undefined :: α

This is a limitation of the computational model

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Two problems with this approach:

- We can't express predicates
   This is a limitation of the type system
   Agda uses a dependent type system
- undefined :: α
   This is a limitation of the computational model In Agda, definitions are total

The C-H correspondence generalizes to other type systems and other logic systems.

To give more precise specifications to our definitions, we need something that corresponds (via the C-H isomorphism) to a more expressive logic.

In Haskell...

► Terms can depend on terms: regular function definitions

- ► Types can depend on types: type constructors like Maybe : ★ → ★
- Terms can depend on types: polymorphism (parametric/typeclasses)

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- ► Types can depend on types: type constructors like Maybe : ★ → ★
- Terms can depend on types: polymorphism (parametric/typeclasses)

So what about *types depending on terms*? This would correspond, via C-H, to predicates. A *dependent type system* is one where types can depend on terms.

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In a dependently-typed setting, the type construction schema  $\Pi$  generalizes the notion of function types, so that the *type of the result depends on the value of the argument*:

$$\frac{\Gamma \vdash A : \star \qquad \Gamma, x : A \vdash B : \star}{\Gamma \vdash \Pi x : A.B : \star}$$
$$\frac{\Gamma \vdash A : \star \qquad \Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x : A.e : \Pi x : A.B}$$
$$\frac{\Gamma \vdash f : \Pi x : A.B \qquad \Gamma \vdash e : A}{\Gamma \vdash f : e : B[e/x]}$$

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The type construction schema  $\Sigma$  generalizes the notion of product types, so that the type of the second coordinate depends on the value of the first coordinate:

$$\frac{\Gamma \vdash A : \star \qquad \Gamma, x : A \vdash B : \star}{\Gamma \vdash \Sigma x : A.B : \star}$$

$$\frac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B[e_1/x]}{\Gamma \vdash (e_1, e_2) : \Sigma x : A.B}$$

$$\frac{\Gamma \vdash e: \Sigma x : A.B}{\Gamma \vdash proj_1 \ e : A}$$

$$\frac{\Gamma \vdash e : \Sigma x : A.B}{\Gamma \vdash proj_2 \ e : B[proj_1 \ e/x]}$$

# Part II

## A taste of Agda

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To understand the following slides, we need to know about a couple of important syntactic distinctions between Haskell and Agda:

Implicit arguments: enclosed between { } symbols

 $map : \{A B : Set\} \rightarrow (A \rightarrow B) \rightarrow List A \rightarrow List B$ 

• Arguments with inferred types: prefixed with  $\forall$ 

 $map : \forall \{A B\} \rightarrow (A \rightarrow B) \rightarrow List A \rightarrow List B$ 

Mixfix notation & unicode characters:

 $_{-}+_{-}:\ \mathbb{N}\rightarrow\mathbb{N}\rightarrow\mathbb{N}$ 

It seems every introduction to Agda aimed at programmers has to start with vectors...

```
data Nat : Set where

zero : Nat

suc : Nat \rightarrow Nat

data Vec (A : Set) : Nat \rightarrow Set where

nil : Vec A zero

cons : (n : Nat) \rightarrow A \rightarrow Vec A n \rightarrow Vec A (suc n)
```

The type of a vector contains its length (a value of type Nat)

It seems every introduction to Agda aimed at programmers has to start with vectors...

```
data \mathbb{N} : Set where

zero : \mathbb{N}

suc : \mathbb{N} \to \mathbb{N}

data Vec (A : Set) : \mathbb{N} \to Set where

[] : Vec A zero

_::_ : \forall \{n\} \to A \to Vec A n \to Vec A (suc n)
```

The type of a vector contains its length (a value of type  $\mathbb{N}$ )

#### map for vectors

With just these definitions, we can already give a richer specification of *map*: one that records the fact that it preserves length.

$$\begin{array}{ll} map : \forall \{A B n\} \rightarrow (A \rightarrow B) \rightarrow Vec \ A n \rightarrow Vec \ B n \\ map \ f \ [] &= \ [] \\ map \ f \ (x \ :: \ xs) \ = \ f \ x \ :: \ map \ f \ xs \end{array}$$

If instead, we wrote

```
map f \_ = []
```

we would get a type error:

```
zero != .n of type \mathbb N when checking that the expression [] has type Vec .B .n
```

{-# LANGUAGE GADTs, DataKinds #-} data Nat = Z | S Nat data Vec a n where Nil :: Vec a Z Cons ::  $a \rightarrow Vec a n \rightarrow Vec a (S n)$ vmap ::  $(a \rightarrow b) \rightarrow Vec a n \rightarrow Vec b n$ vmap f Nil = Nil vmap f (Cons x xs) = Cons (f x) \$ vmap f xs

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<sup>1</sup>and data kinds

The power of  $\Pi$  types is that you can lift arbitrary terms into your types, not just (types representing lifted) constructors. E.g. if we have:

then we can also write:

$$\begin{array}{ll} - + + - &: \forall \{A \ n \ m\} \rightarrow Vec \ A \ n \rightarrow Vec \ A \ m \rightarrow Vec \ A \ (n + m) \\ [] & + ys = ys \\ (x \ :: \ xs) \ + \ ys = x \ :: \ (xs \ + \ ys) \end{array}$$

Just like with GADTs, when you pattern match on e.g. [], locally (for the right-hand side) the type of  $_{-}$  ++  $_{-}$  is specialized to

 $_{-}$  ++  $_{-}$  :  $\forall$  {A m}  $\rightarrow$  Vec A zero  $\rightarrow$  Vec A m  $\rightarrow$  Vec A (zero + m)

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On the other hand, the type of the right-hand side is:

ys : Vec A m

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On the other hand, the type of the right-hand side is:

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When this right-hand side is typechecked, it has to reduce the function application zero + m to m at compile type. That's the magic sauce.

If we, instead, wrote

 $\_++\_: \forall \{A \ n \ m\} \rightarrow Vec \ A \ n \rightarrow Vec \ A \ m \rightarrow Vec \ A \ (m + n),$ 

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then the typechecker would reject the same definition, because e.g. for the first branch of *append* setting *n* to *zero*, *zero* + *m* and m + zero are not the same terms: the first one reduces to *m*, whereas the second one cannot be reduced further without knowing anything about *m*.

We can define our own equality relation by reflexivity:

data \_ = \_ { A : Set } : A 
$$\rightarrow$$
 A  $\rightarrow$  Set where  
refl :  $\forall$  {x}  $\rightarrow$  x = x

When we pattern match on *refl*, we learn about other arguments as well. That's why we can prove the following congruence:

$$cong : \forall \{A B x y\} \rightarrow (f : A \rightarrow B) \rightarrow x \equiv y \rightarrow f x \equiv f y$$
  
$$cong f refl = refl$$

since by matching refl, the type for that branch becomes

$$cong : \forall \{A B x . x\} \rightarrow (f : A \rightarrow B) \rightarrow x \equiv .x \rightarrow f x \equiv f x$$

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Proofs about equalities simpy encode the needed equality in their types. So let's try to prove something:

$$a_{-} + 0 : \forall n \rightarrow n \equiv (n + zero)$$
  
 $n + 0 = refl$ 

Of course, this will be rejected by the type checker, since n + zero and n are not the same terms, and neither can be reduced further. To reduce n + zero, we need to know about n's constructor:

$$_{-}+0$$
:  $\forall n \rightarrow n \equiv (n + zero)$   
zero  $+0 = refl$   
suc  $n + 0 = cong suc (n + 0)$ 

#### Proving equalities: + is commutative

To get a better feel of these proofs, let's prove that  $+ \mbox{ is commutative:}$ 

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$$+- comm$$
 :  $\forall n m \rightarrow (n+m) \equiv (m+n)$ 

Let's consider each of the four cases separately:

• 
$$0 + 0 \equiv 0 + 0$$
: Both sides reduce to 0

+- comm zero zero = refl

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+- comm zero zero = refl

 0 + Sm ≡ Sm + 0: Since 0 + Sm → Sm and Sn + m → S (n + m), we can recurse by taking the suc of both sides:

+- comm zero (suc m) = cong suc (+- comm zero m)

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•  $Sn + 0 \equiv 0 + Sn$ : Analogous to the previous one:

+ - comm (suc n) zero = cong suc (+ - comm n zero)

# Proving equalities: + is commutative (cont.)

We are left with the fourth case:  $Sn + Sm \equiv Sm + Sn$ . To prove that, we will need a property of equality (transitivity) and a lemma about +.

$$\begin{array}{l} \text{infixl } 10 \ _\langle trans \rangle_- \\ \_\langle trans \rangle_- : \ \forall \ \{A\} \ \{x \ y \ z \ : \ A\} \rightarrow x \equiv y \rightarrow y \equiv z \rightarrow x \equiv z \\ \text{refl} \ \langle trans \rangle \ \text{refl} \ = \ \text{refl} \end{array}$$

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$$\begin{array}{rl} +-\operatorname{comm}(\operatorname{suc} n)(\operatorname{suc} m) &= \operatorname{cong}\operatorname{suc}(\\ +-\operatorname{comm} n(\operatorname{suc} m)\\ \langle \operatorname{trans}\rangle\\ \operatorname{cong}\operatorname{suc}(+-\operatorname{comm} m n)\\ \langle \operatorname{trans}\rangle\\ +-\operatorname{comm}(\operatorname{suc} n)m\\ \end{array}$$

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The previous proof is basically unreadable...

Fortunately, the standard library has a couple of combinators to make equality proofs read like informal ones:

$$-- comm (suc n) (suc m) = cong suc $$$

$$begin$$

$$n + suc m \equiv \langle + - comm n (suc m) \rangle$$

$$suc m + n \equiv \langle cong suc (+ - comm m n) \rangle$$

$$suc n + m \equiv \langle + - comm (suc n) m \rangle$$

$$m + suc n$$

$$\Box$$

Now that we have proven that  $n + m \equiv m + n$ , we can use the equality to substitute one for the other in types:

$$\begin{aligned} \text{subst} &: \{A : Set\} \rightarrow (P : A \rightarrow Set) \\ &\rightarrow \forall \{x \ y\} \rightarrow x \equiv y \\ &\rightarrow P \ x \rightarrow P \ y \\ \\ \text{subst} \ P \ \text{refl} \ \text{prf} \ = \ \text{prf} \end{aligned}$$

Which allows us to write:

$$\begin{array}{l} ++'_{-}: \forall \{A \ n \ m\} \rightarrow Vec \ A \ n \rightarrow Vec \ A \ m \rightarrow Vec \ A \ (m+n) \\ ++'_{-} \{n = n\} \ \{m = m\} \ xs \ ys = \\ subst \ (Vec \ \_) \ (+ - comm \ n \ m) \ (xs \ + \ ys) \end{array}$$

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# Part III

## The MU Puzzle

In *Gödel, Escher, Bach*, Hofstadter describes a very simple string rewriting system with the following rules:

- MI is a valid string
- ► You can append a *U* to any valid string ending with *I*
- You can double the string after the initial M
- ► Any *III* can be replaced with a single *U*
- ► Any occurances of *UU* can be removed

Hofstadter then asks whether it's possible to derive MU from MI using these rules.

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#### module MU where

Since strings of the MU system always start with an M and contain only I and U afterwards, we can represent words as such:

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data Symbol : Set where
 I : Symbol
 U : Symbol
open import Data.List
Word : Set
Word = List Symbol

We can transliterate the rules into a datatype, where each constructor corresponds to one of the derivation rules. The type is indexed by the word that results from that particular sequence of derivation steps.

data M : Word  $\rightarrow$  Set where MI : *M*[/]  $M \times I \rightarrow M \times I U : \forall \{x\} \rightarrow M (x + I :: []) \rightarrow$ M(x + I :: U :: []) $Mx \rightarrow Mxx$  :  $\forall \{x\} \rightarrow M x \rightarrow$ M(x + x) $III \rightarrow U \qquad : \forall \{x y\} \rightarrow M (x + I :: I :: I :: y) \rightarrow$ M(x + U :: y) $UU \rightarrow \varepsilon$  :  $\forall \{x y\} \rightarrow M(x + U :: U :: y) \rightarrow$ M(x + y)

We can use this definition to prove that e.g. *MIUIU* is a valid string:

Note that we had to help Agda a bit when applying  $MxI \rightarrow MxIU$ , since it cannot automatically determine that if x + I :: [] = I :: [], then x = [].

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It can be proven that MU is not a valid string, using the invariant that the number of I characters in every valid string is not divisible by 3. Since the number of I's in MU is 0, and 0 is trivially divisible by 3, we can conclude that MU is not a valid string. How can we write such a proof in Agda?

#### Negation

So far, every proposition was a positive one, and every proof has been constructive. How can we encode negation and proof by contradiction into this system?

### Negation

So far, every proposition was a positive one, and every proof has been constructive. How can we encode negation and proof by contradiction into this system?

By using an *absurd type* to denote false statements, and giving an elimination rule that encodes *ex falso quodlibet*:

data  $\perp$  : Set where

$$\neg_{-} : Set \rightarrow Set$$
  

$$\neg A = A \rightarrow \bot$$
  

$$\bot - elim : \forall \{P : Set\} \rightarrow \bot \rightarrow P$$
  

$$\bot - elim ()$$

This works because Agda knows there is no pattern that can match  $\perp$ . It also means we can't introduce values of type  $\perp$  without matching on some other absurd pattern.

In some logic systems, the following is true:

excluded – middle :  $\{A B : Set\} \rightarrow \neg \neg A \rightarrow A$ 

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However, the proof scheme this encodes is necessarily non-constructive.

In some logic systems, the following is true:

```
excluded – middle : \{A B : Set\} \rightarrow \neg \neg A \rightarrow A
```

However, the proof scheme this encodes is necessarily non-constructive.

In Agda, we cannot prove this.

#### Proving *MI* is not a valid word

Our goal is to prove the following proposition:

 $\neg MU : \neg M[U]$ 

and our plan is to do it via the following invariant, which we'll prove inductively:

#### open import Data.Nat

$$#I : Word \rightarrow \mathbb{N}$$

$$#I[] = 0$$

$$#I(I :: x) = suc(#Ix)$$

$$#I(U :: x) = #Ix$$

open import Data.Nat.Divisibility

invariant :  $\forall \{x\} \to M x \to \text{Invariant } x \to ( \mathbb{R} ) \to ( \mathbb{R} ) \to ( \mathbb{R} )$ 

The type  $_{-}|_{-}$  we use in the declaration of *inv* comes from the standard library, and is defined as the following:

data \_ | \_ : 
$$\mathbb{N} \to \mathbb{N} \to Set$$
 where  
divides : {  $m n : \mathbb{N}$  } ( $q : \mathbb{N}$ ) ( $eq : n \equiv q * m$ )  $\to m \mid n$ 

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# A couple of proofs about #I

See the full code for the definitions; for now, it's enough to understand the statements themselves.

$$#I - ++ : \forall x y \rightarrow$$
$$#I (x + y) \equiv #I x + #I y$$

$$\#I - xIU : \forall x \to \\ \#I (x + I :: U :: []) \equiv \#I (x + I :: [])$$

$$\#I - xUy : \forall x y \rightarrow \#I (x + y) \equiv \#I (x + U :: y)$$

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To prove the base case, we only need to prove  $3 \nmid 1$ , which we can do by trying to pattern-match on the equation inside *divides*, and realizing it cannot hold:

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```
invariant : \forall \{x\} \rightarrow M \ x \rightarrow \text{Invariant } x
invariant MI = 3 \nmid 1
where
3 \nmid 1 : 3 \nmid 1
3 \nmid 1 (divides zero ())
3 \nmid 1 (divides (suc q) ())
```

#### Proving the invariant: Induction

Using the properties of #I we proved earlier, it's easy to use induction to prove some of the other cases:

 $invariant (MxI \rightarrow MxIU \{x\} MxI)$   $= invariant MxI \circ keep (#I - xIU x)$   $invariant (UU \rightarrow \varepsilon \{x\} \{y\} MxUUy)$   $= invariant MxUUy \circ keep lemma$ where  $lemma : #I (x + y) \equiv #I (x + U :: U :: y)$ 

 $lemma = \#I - xUy \times y \langle trans \rangle \#I - xUy \times (U :: y)$ 

#### Proving the invariant: Induction

Using the properties of #I we proved earlier, it's easy to use induction to prove some of the other cases:

 $\begin{array}{l} \text{invariant } (M \times I \to M \times IU \{x\} M \times I) \\ = \text{ invariant } M \times I \circ \text{keep } (\#I - xIU \times) \\ \text{invariant } (UU \to \varepsilon \{x\} \{y\} M \times UUy) \\ = \text{ invariant } M \times UUy \circ \text{keep lemma} \\ \textbf{where} \\ \text{lemma } : \#I (x + y) \equiv \#I (x + U :: U :: y) \\ \text{lemma } = \#I - xUy \times y \langle \text{ trans } \rangle \#I - xUy \times (U :: y) \end{array}$ 

So the tricky ones that remain are:

 $\begin{array}{l} \text{invariant (III} \rightarrow U \{x\} \{y\} MxIIIy) \\ = \text{ invariant } MxIIIy \circ ? \\ & -- \text{ Need a proof that if } 3 \mid \#I \; xUy, \text{ then } 3 \mid \#I \; xIIIy \\ \text{invariant } (Mx \rightarrow Mxx \{x\} Mx) \\ = \text{ invariant } Mx \circ keep ? \\ & -- \text{ Need a proof that if } 3 \mid x, \text{ then } 3 \mid x + x \\ & = \text{ invariant } \mathbb{R} \times \mathbb{R} \xrightarrow{\mathbb{R}} \mathbb{R} \times \mathbb{R} \xrightarrow{\mathbb{R}} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}} \xrightarrow{\mathbb{R}} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}} \xrightarrow{\mathbb{R$ 

First of all, we know  $#I \times Uy \equiv #I \times y$ , and also that  $#I \times IIIy \equiv 3 + #I \times y$ , so the important lemma is that  $3 \mid x \rightarrow 3 \mid 3 + x$ :

invariant (III 
$$\rightarrow U \{x\} \{y\} MxIIIy$$
)  
= invariant MxIIIy  $\circ$  proof  
where

 $lemma_1 : \forall n \rightarrow 3 \mid n \rightarrow 3 \mid 3 + n$ 

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First of all, we know  $#I \times Uy \equiv #I \times y$ , and also that  $#I \times IIIy \equiv 3 + #I \times y$ , so the important lemma is that  $3 \mid x \rightarrow 3 \mid 3 + x$ :

invariant (III 
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)  
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where

$$\textit{lemma}_1 \ : \ \forall \ n \to 3 \mid n \to 3 \mid 3 + n$$

invariant (III 
$$\rightarrow U \{x\} \{y\} MxIIIy$$
)  
= invariant MxIIIy  $\circ$  proof  
where

$$lemma_2 : 3 + \#I(x + U :: y) \equiv \\ \#I(x + I :: I :: I :: y)$$

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invariant (III 
$$\rightarrow U \{x\} \{y\} MxIIIy$$
)  
= invariant MxIIIy  $\circ$  proof  
where

$$lemma_2 : 3 + #I(x + U :: y) \equiv #I(x + I :: I :: I :: y)$$

lemma<sub>2</sub>

$$= cong (_+ +_- 3) (sym (#I - xUy \times y)) \langle trans \rangle #I - xIIIy x y$$

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 $3 \mid \#I \times Uy \rightarrow 3 \mid \#I \times IIIy$ 

invariant (III 
$$\rightarrow U \{x\} \{y\} MxIIIy$$
)  
= invariant MxIIIy  $\circ$  proof  
where

$$lemma_1 : \forall n \to 3 \mid n \to 3 \mid 3 + n$$

$$lemma_2 : 3 + \#I (x + U :: y) \equiv \\ \#I (x + I :: I :: I :: y)$$

proof : 
$$3 | #I (x + U :: y) \rightarrow$$
  
 $3 | #I (x + I :: I :: I :: y)$   
proof

= keep lemma<sub>2</sub>  $\circ$  lemma<sub>1</sub> (#I (x + U :: y))

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For the indirect proof here, the crucial lemma is that if  $3 \mid 2 * n$ , then  $3 \mid n$  would also hold, which is in contradiction with our inductive assumption.

invariant 
$$(Mx \rightarrow Mxx \{x\} Mx)$$
  
= invariant  $Mx \circ lemma \circ keep (\#I - dup x)$   
where  
 $dup : \forall n \rightarrow n + n \equiv 2 * n$   
 $dup n = cong (\_+\_n) (n+0)$   
 $\#I - dup : \forall x \rightarrow \#I (x + x) \equiv 2 * \#I x$   
 $\#I - dup x = \#I - ++ x x \langle trans \rangle dup (\#I x)$   
 $lemma : \forall \{n\} \rightarrow 3 \mid 2 * n \rightarrow 3 \mid n$ 

# $3 \mid 2 * n \rightarrow 3 \mid n$

The standard library contains definitions and proofs of some pretty high-level stuff, so we can prove  $3 \mid 2 * n \rightarrow 3 \mid n$  by observing that 2 and 3 are co-primes...

*lemma* :  $\forall \{n\} \rightarrow 3 \mid 2 * n \rightarrow 3 \mid n$ lemma = coprime - divisor 3 - coprime - 2where **open import** *Data*.*Nat*.*Coprimality* 3 - coprime - 2 : Coprime 3 2 $3 - coprime - 2 = prime \Rightarrow coprime - 3 - prime 2$  $(from - ves (1 \leq ? 2))$ (from  $-yes (3 \leq ? 3)$ ) where **open import** Data.Nat.Primality open import Relation.Nullary.Decidable 3 - prime : Prime 3 3 - prime = from - yes (prime? 3) - + (B) + (B)

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Stratified universes Totality and the termination checker Coinductive types & corecursive definitions

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Stratified universes Totality and the termination checker Coinductive types & corecursive definitions And lot more...

# Questions?